The Tetrahedron Algebra and its Finite-Dimensional Irreducible Modules

Brian Hartwig *

Abstract

Recently Terwilliger and the present author found a presentation for the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. To obtain this presentation we defined a Lie algebra $\boxtimes$ by generators and relations and displayed an isomorphism from $\boxtimes$ to the three-point $\mathfrak{sl}_2$ loop algebra. In this paper we classify the finite-dimensional irreducible $\boxtimes$-modules.

1 Introduction and Statement of Results

In [8] Terwilliger and the present author gave a presentation of the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. To obtain this presentation we defined a Lie algebra $\boxtimes$ by generators and relations and displayed an isomorphism from $\boxtimes$ to the three-point $\mathfrak{sl}_2$ loop algebra.

In this paper we classify the finite-dimensional irreducible $\boxtimes$-modules. To obtain this classification, we exploit a connection between $\boxtimes$ and a certain infinite-dimensional Lie algebra which appears in the physics literature, called the Onsager algebra [1], [6], [7], [12]. Before we explain our results we first give some background on $\boxtimes$ and the Onsager algebra. We start with their definitions.

Throughout the paper $\mathbb{K}$ denotes an algebraically closed field of characteristic 0.

Definition 1.1 [8] Let $\boxtimes$ denote the Lie algebra over $\mathbb{K}$ with generators

$$\{X_{rs} | r, s \in \mathbb{I}, r \neq s\} \qquad\qquad \mathbb{I} = \{0, 1, 2, 3\}$$

and the following relations.

(i) For all distinct $r, s \in \mathbb{I}$,

$$X_{rs} + X_{sr} = 0. \tag{1}$$

(ii) For all mutually distinct $r, s, t \in \mathbb{I}$,

$$[X_{rs}, X_{st}] = 2X_{rs} + 2X_{st}. \tag{2}$$

*Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison WI 53706-1388 USA

(iii) For all mutually distinct $r, s, t, u \in \mathbb{I}$,

$$[X_{rs}, [X_{rs}, [X_{rs}, X_{tu}]]] = 4[X_{rs}, X_{tu}]. \tag{3}$$

We call $\boxtimes$ the *tetrahedron algebra*.

Definition 1.2 [12], [13] Let $\mathcal{O}$ denote the Lie algebra over $\mathbb{K}$ with generators X, Y and relations

$$[X, [X, [X, Y]]] = 4[X, Y], \tag{4}$$
$$[Y, [Y, [Y, X]]] = 4[Y, X]. \tag{5}$$

We call $\mathcal{O}$ the *Onsager algebra*. We call X, Y the *standard generators* for $\mathcal{O}$.

We recall the connection between $\boxtimes$ and $\mathcal{O}$. For mutually distinct $r, s, t, u \in \mathbb{I}$ there exists a Lie algebra injection from $\mathcal{O}$ into $\boxtimes$ that sends

$$\begin{aligned} X &\mapsto X_{rs} \\ Y &\mapsto X_{tu} \end{aligned}$$

[8, Corollary 12.2]. We call the image of this injection an *Onsager subalgebra* of $\boxtimes$. Observe that $\boxtimes$ has three Onsager subalgebras. By [8, Theorem 11.6] the $\mathbb{K}$-vector space $\boxtimes$ is the direct sum of its three Onsager subalgebras.

We now summarize the classification of finite-dimensional irreducible $\mathcal{O}$-modules [4], [5]. We begin with a comment. Let V denote a finite-dimensional irreducible $\mathcal{O}$-module. As we will see in Section 2, the standard generators X, Y are diagonalizable on V. Furthermore there exist an integer $d \geq 0$ and scalars $\alpha, \alpha^* \in \mathbb{K}$ such that the set of distinct eigenvalues of X (resp. Y) on V is $\{d - 2i + \alpha | 0 \leq i \leq d\}$ (resp. $\{d - 2i + \alpha^* | 0 \leq i \leq d\}$). We call the ordered pair (α, α^*) the *type* of V. Replacing X, Y by $X - \alpha I, Y - \alpha^* I$ the type becomes $(0, 0)$. Therefore it suffices to classify the finite-dimensional irreducible $\mathcal{O}$-modules of type $(0, 0)$.

We begin with a special case. Observe that up to isomorphism, there exists a unique irreducible $\mathcal{O}$-module of dimension 1 and type $(0, 0)$. We call this the *trivial* $\mathcal{O}$-module.

Let $\mathfrak{sl}_2$ denote the Lie algebra over $\mathbb{K}$ with basis e, f, h and Lie bracket

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

Let t denote an indeterminate and let $L(\mathfrak{sl}_2)$ denote the loop algebra $\mathfrak{sl}_2 \otimes \mathbb{K}[t, t^{-1}]$, where $\otimes$ means $\otimes_{\mathbb{K}}$. By [14, Proposition 1] there exists a Lie algebra injection from $\mathcal{O}$ into $L(\mathfrak{sl}_2)$ that sends

$$\begin{aligned} X &\mapsto e \otimes 1 + f \otimes 1 \\ Y &\mapsto e \otimes t + f \otimes t^{-1}. \end{aligned}$$

For the moment we identify $\mathcal{O}$ with its image under the above injection. For nonzero $a \in \mathbb{K}$ we define the Lie algebra homomorphism $EV_a : L(\mathfrak{sl}_2) \to \mathfrak{sl}_2$ by $EV_a(u \otimes g(t)) = g(a)u$ for all $u \in \mathfrak{sl}_2$ and $g(t) \in \mathbb{K}[t, t^{-1}]$. Let ev_a denote the restriction of EV_a to $\mathcal{O}$. Then $ev_a : \mathcal{O} \to \mathfrak{sl}_2$

is a Lie algebra homomorphism which we call the *evaluation homomorphism* for a. Let V denote an irreducible $\mathfrak{sl}_2$-module with finite dimension at least 2. We pull back by ev_a to get an $\mathcal{O}$-module structure on V. We call this an *evaluation module* for $\mathcal{O}$ and denote it by $V(a)$. The $\mathcal{O}$-module $V(a)$ is irreducible if and only if $a \neq \pm 1$ [3, Lemma 4]. Irreducible $\mathcal{O}$-modules $V(a)$ and $V(b)$ are isomorphic if and only if $a = b^{\pm 1}$ [14, Proposition 5].

Let U, V denote $\mathcal{O}$-modules. Then $U \otimes V$ has an $\mathcal{O}$-module structure given by

$$x.(u \otimes v) = (x.u) \otimes v + u \otimes (x.v) \qquad x \in \mathcal{O}, \quad u \in U, \quad v \in V.$$

Let V denote an $\mathcal{O}$-module that is the tensor product of finitely many evaluation modules. If V is irreducible then it is type $(0,0)$ [3, p. 3281].

The classification of finite-dimensional irreducible $\mathcal{O}$-modules of type $(0,0)$ is given in the following three theorems.

Theorem 1.3 *[3, Theorem 6] Every nontrivial finite-dimensional irreducible $\mathcal{O}$-module of type $(0,0)$ is isomorphic to a tensor product of evaluation modules.*

Theorem 1.4 *[3, Proposition 5] Let $V_1(a_1), \ldots, V_n(a_n)$ denote a finite sequence of evaluation modules for $\mathcal{O}$, and consider the $\mathcal{O}$-module $V_1(a_1) \otimes \cdots \otimes V_n(a_n)$. This module is irreducible if and only if $a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}$ are mutually distinct.*

Definition 1.5 Let $V_1(a_1), \ldots, V_n(a_n)$ denote a finite sequence of evaluation modules for $\mathcal{O}$. Let V denote the $\mathcal{O}$-module $V_1(a_1) \otimes \cdots \otimes V_n(a_n)$. Any tensor product of evaluation modules that can be obtained from V by permuting the order of the factors and replacing any number of the a_i's with their multiplicative inverses will be called *equivalent* to V.

Theorem 1.6 *[3, Proposition 5] Let U and V denote tensor products of finitely many evaluation modules for $\mathcal{O}$. Assume each of U, V is irreducible as an $\mathcal{O}$-module. Then the $\mathcal{O}$-modules U and V are isomorphic if and only if they are equivalent.*

This completes the classification of finite-dimensional irreducible $\mathcal{O}$-modules.

We now state our main results. They are contained in the following two theorems and subsequent remark.

Theorem 1.7 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Then there exists a unique $\mathcal{O}$-module structure on V such that the standard generators X, Y act on V as X_{01}, X_{23} respectively. This $\mathcal{O}$-module structure is irreducible and has type $(0,0)$.*

Theorem 1.8 *Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0,0)$. Then there exists a unique $\boxtimes$-module structure on V such that the standard generators X, Y act on V as X_{01}, X_{23} respectively. This $\boxtimes$-module structure is irreducible.*

Remark 1.9 *Combining the previous two theorems we obtain a bijection between the following two sets:*

(i) The isomorphism classes of finite-dimensional irreducible $\mathcal{O}$-modules of type $(0,0)$.

(ii) The isomorphism classes of finite-dimensional irreducible $\boxtimes$-modules.

2 $\mathcal{O}$-Modules and Tridiagonal Pairs

In order to prove Theorem 1.7 and Theorem 1.8 it will be useful to consider how finite-dimensional irreducible $\mathcal{O}$-modules are related to tridiagonal pairs. To explain this relationship we use the following concepts. Let V denote a vector space over $\mathbb{K}$ with positive finite dimension. By a *linear transformation on* V we mean a $\mathbb{K}$-linear map from V to V. Let A denote a linear transformation on V. For $\lambda \in \mathbb{K}$ define

$$V_A(\lambda) := \{v \in V | Av = \lambda v\}. \tag{6}$$

Observe that λ is an eigenvalue for A if and only if $V_A(\lambda) \neq 0$, and in this case $V_A(\lambda)$ is the corresponding eigenspace of A. We recall that the sum $\sum_{\lambda\in\mathbb{K}} V_A(\lambda)$ is direct. We call A *diagonalizable* whenever $V = \sum_{\lambda\in\mathbb{K}} V_A(\lambda)$.

Lemma 2.1 *Let V denote a vector space over $\mathbb{K}$ with positive finite dimension. Let A and A^* denote linear transformations on V. Then for $\lambda \in \mathbb{K}$ the following $(i), (ii)$ are equivalent.*

(i) The expression $[A, [A, [A, A^*]]] - 4[A, A^*]$ vanishes on $V_A(\lambda)$.

(ii)

$$A^*V_A(\lambda) \subseteq V_A(\lambda + 2) + V_A(\lambda) + V_A(\lambda - 2). \tag{7}$$

Proof. Let Φ denote the expression in (i) and observe

$$\Phi = A^3A^* - 3A^2A^*A + 3AA^*A^2 - A^*A^3 - 4AA^* + 4A^*A.$$

For $v \in V_A(\lambda)$ we evaluate Φv using $Av = \lambda v$ to find

$$\begin{aligned} \Phi v &= (A^3A^* - 3\lambda A^2A^* + 3\lambda^2 AA^* - \lambda^3 A^* - 4AA^* + 4\lambda A^*)v \\ &= (A - (\lambda + 2)I)(A - \lambda I)(A - (\lambda - 2)I)A^*v. \end{aligned}$$

The scalars $\lambda + 2, \lambda, \lambda - 2$ are mutually distinct since $\mathrm{Char}(\mathbb{K}) = 0$. The result follows. ∎

We now recall the concept of a *tridiagonal pair* [2], [9], [10], [11].

Definition 2.2 [10] Let V denote a vector space over $\mathbb{K}$ with positive finite dimension. By a *tridiagonal pair* on V, we mean an ordered pair A, A^* where A and A^* are linear transformations on V that satisfy the following four conditions.

(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$A^*V_i \subseteq V_{i+1} + V_i + V_{i-1} \qquad 0 \leq i \leq d, \tag{8}$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i+1}^* + V_i^* + V_{i-1}^* \qquad\qquad 0 \leq i \leq \delta,$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, and $W \neq V$.

We comment on Definition 2.2. Let A, A^* denote a tridiagonal pair on V and let d and δ be as in Definition 2.2. From [10, Lemma 4.5] we find $d = \delta$. We call this common value the *diameter* of A, A^*. An ordering of the eigenspaces of A (resp. A^*) will be called *standard* whenever it satisfies Definition 2.2(ii) (resp. Definition 2.2(iii)). Let $V_0, \ldots, V_d$ denote a standard ordering of the eigenspaces of A. By [15, p. 388], the ordering $V_d, \ldots, V_0$ is standard and no other ordering is standard. A similar result holds for A^*. By an *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of A, A^* we mean an ordering of the eigenvalues for A (resp. A^*) for which the corresponding ordering of the eigenspaces is standard.

Definition 2.3 Let $\theta_0, \theta_1, \ldots, \theta_d$ denote a finite sequence of scalars in $\mathbb{K}$. We say this sequence is *arithmetic with common difference* 2 whenever $\theta_{i-1} - \theta_i = 2$ for $1 \leq i \leq d$.

Theorem 2.4 *Let V denote a vector space over $\mathbb{K}$ with positive finite dimension. Let A and A^* denote linear transformations on V. Then the following $(i), (ii)$ are equivalent.*

(i) A, A^* is a tridiagonal pair on V whose eiqenvalue sequence and dual eigenvalue sequence are both arithmetic with common difference 2.

(ii) There exists an irreducible $\mathcal{O}$-module structure on V such that the standard generators X, Y act on V as A, A^* respectively.

Proof. $(i) \Rightarrow (ii)$: We first show that A, A^* satisfy

$$[A, [A, [A, A^*]]] = 4[A, A^*]. \tag{9}$$

By assumption the tridiagonal pair A, A^* has an eigenvalue sequence that is arithmetic with common difference 2. Let $V_0, V_1, \ldots, V_d$ denote the corresponding ordering of the eigenspaces of A. This ordering is standard by construction so it satisfies (8). By these comments we find that A^* satisfies (7) for all $\lambda \in \mathbb{K}$. By Lemma 2.1 and since A is diagonalizable on V we find A, A^* satisfy (9). Reversing the roles of A and A^* in the above argument we find A, A^* satisfy

$$[A^*, [A^*, [A^*, A]]] = 4[A^*, A]. \tag{10}$$

By (9) and (10) there exists an $\mathcal{O}$-module structure on V such that X, Y act as A, A^* respectively. This $\mathcal{O}$-modules structure is irreducible by Definition 2.2(iv).

$(ii) \Rightarrow (i)$: [10, Example 1.6] Since $\mathbb{K}$ is algebraically closed all the eigenvalues for A are contained in $\mathbb{K}$. Since V has positive finite dimension, A has at least one eigenvalue λ. Since $\mathrm{Char}(\mathbb{K}) = 0$, the scalars $\lambda, \lambda + 2, \ldots$ are mutually distinct and therefore cannot all

be eigenvalues for A. Hence there exists an eigenvalue θ for A such that $\theta + 2$ is not an eigenvalue for A. The scalars $\theta, \theta - 2, \ldots$ are mutually distinct and therefore cannot all be eigenvalues for A. Hence there exists a nonnegative integer d such that $\theta - 2i$ is an eigenvalue of A for $0 \leq i \leq d$ but is not an eigenvalue of A for $i = d+1$. Abbreviate $V_A(\theta - 2i)$ by V_i for $0 \leq i \leq d$. By construction $\sum_{i=0}^{d} V_i$ is A-invariant. By Lemma 2.1 we find

$$A^* V_i \subseteq V_{i+1} + V_i + V_{i-1} \qquad (0 \leq i \leq d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$. Therefore $\sum_{i=0}^{d} V_i$ is A^*-invariant. Since the $\mathcal{O}$-module structure is irreducible and since $\sum_{i=0}^{d} V_i \neq 0$ we find $\sum_{i=0}^{d} V_i = V$. We have now shown that A is diagonalizable and Definition 2.2(ii) holds. Reversing the roles of A and A^* in the above argument we find A^* is diagonalizable and Definition 2.2(iii) holds. Definition 2.2(iv) is immediate since the $\mathcal{O}$-module V is irreducible. We have now shown that A, A^* is a tridiagonal pair on V. Recall that $\theta - 2i$ is the eigenvalue for A associated with V_i for $0 \leq i \leq d$. The ordering $V_0, \ldots, V_d$ is standard so the sequence $\theta - 2i$ $(0 \leq i \leq d)$ is an eigenvalue sequence for A, A^*. This sequence is arithmetic with common difference 2. We have now shown A, A^* has an arithmetic eigenvalue sequence with common difference 2. Reversing the roles of A and A^* in the above argument we find that A, A^* has an arithmetic dual eigenvalue sequence with common difference 2. ∎

Definition 2.5 Let V denote a finite-dimensional irreducible $\mathcal{O}$-module. By Theorem 2.4 there exist an integer $d \geq 0$ and scalars $\alpha, \alpha^* \in \mathbb{K}$ such that the set of distinct eigenvalues of X (resp. Y) on V is $\{d - 2i + \alpha | 0 \leq i \leq d\}$ (resp. $\{d - 2i + \alpha^* | 0 \leq i \leq d\}$). We call d the *diameter* of V. We call the ordered pair (α, α^*) the *type* of V.

Note 2.6 Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type (α, α^*). Replacing X, Y by $X - \alpha I, Y - \alpha^* I$ the type becomes $(0, 0)$.

Restricting Theorem 2.4 to type $(0, 0)$ we get the following corollary.

Corollary 2.7 *Let V denote a vector space over $\mathbb{K}$ with positive finite dimension. Let A and A^* denote linear transformations on V. Then the following (i), (ii) are equivalent.*

(i) A, A^* is a tridiagonal pair on V and $d - 2i$ $(0 \leq i \leq d)$ is both an eigenvalue sequence and dual eigenvalue sequence for A, A^*, where d denotes the diameter.

(ii) There exists an irreducible $\mathcal{O}$-module structure on V of type $(0, 0)$ such that the standard generators X, Y act on V as A, A^* respectively.

Proof. Immediate from Theorem 2.4 and the definition of type.

3 Finite-Dimensional Irreducible $\boxtimes$-Modules

Let V denote a finite-dimensional irreducible $\boxtimes$-module. In this section we obtain the following description of V. We show that each generator X_{rs} of $\boxtimes$ is diagonalizable on V. We show that there exists an integer $d \geq 0$ such that for each X_{rs} the eigenvalues for X_{rs} on V are $d, d-2, \ldots, -d$. We give the action of each X_{rs} on the eigenspaces of the other generators. In our investigation we use the following lemma.

Lemma 3.1 *Let V denote a vector space over $\mathbb{K}$ with positive finite dimension. Let A and B denote linear transformations on V. Then for $\lambda \in \mathbb{K}$ the following $(i), (ii)$ are equivalent.*

(i) The expression $[A, B] - 2A - 2B$ vanishes on $V_A(\lambda)$.

(ii) $(B + \lambda I)V_A(\lambda) \subseteq V_A(\lambda + 2)$.

Proof. Let Ψ denote the expression in (i) and observe

$$\Psi = AB - BA - 2A - 2B.$$

For $v \in V_A(\lambda)$ we evaluate Ψv using $Av = \lambda v$ and find

$$\Psi v \quad = \quad (A - (\lambda + 2)I)(B + \lambda I)v.$$

The result follows. ∎

We refine notation (6) as follows.

Notation 3.2 With reference to Definition 1.1 let V denote a finite-dimensional irreducible $\boxtimes$-module. For distinct $r, s \in \mathbb{I}$ and for all $\lambda \in \mathbb{K}$ we define

$$V_{rs}(\lambda) = \{v \in V | X_{rs}v = \lambda v\}. \tag{11}$$

Theorem 3.3 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. For all $r, s, t, u \in \mathbb{I}$ $(r \neq s, t \neq u)$ and for all $\lambda \in \mathbb{K}$ the action of X_{tu} on $V_{rs}(\lambda)$ is given as follows.*

Case		*Action of X_{tu} on $V_{rs}(\lambda)$*
$t = r,$	$u = s$	$(X_{tu} - \lambda I)V_{rs}(\lambda) = 0$
$t = s,$	$u = r$	$(X_{tu} + \lambda I)V_{rs}(\lambda) = 0$
$t = s,$	$u \neq r$	$(X_{tu} + \lambda I)V_{rs}(\lambda) \subseteq V_{rs}(\lambda + 2)$
$t \neq r,$	$u = s$	$(X_{tu} - \lambda I)V_{rs}(\lambda) \subseteq V_{rs}(\lambda + 2)$
$t = r,$	$u \neq s$	$(X_{tu} - \lambda I)V_{rs}(\lambda) \subseteq V_{rs}(\lambda - 2)$
$t \neq s,$	$u = r$	$(X_{tu} + \lambda I)V_{rs}(\lambda) \subseteq V_{rs}(\lambda - 2)$
r, s, t, u *distinct*		$X_{tu}V_{rs}(\lambda) \subseteq V_{rs}(\lambda + 2) + V_{rs}(\lambda) + V_{rs}(\lambda - 2)$

We are using Notation 3.2.

Proof. We consider each row in the above table.
$t = r, u = s$: Immediate from (11).
$t = s, u = r$: Immediate from (11) and since $X_{rs} = -X_{tu}$ by Definition 1.1(i).
$t = s, u \neq r$: Combine Definition 1.1(ii) and Lemma 3.1.

$t \neq r, u = s$: By case $t = s, u \neq r$ above and since $X_{ut} = -X_{tu}$.
$t = r, u \neq s$: By case $t = s, u \neq r$ above and since $V_{rs}(\lambda) = V_{sr}(-\lambda)$.
$t \neq s, u = r$: By case $t = r, u \neq s$ above and since $X_{ut} = -X_{tu}$.
r, s, t, u distinct: Combine Definition 1.1(iii) and Lemma 2.1. ∎

Corollary 3.4 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Then for all $r, s, t, u \in \mathbb{I}$ $(r \neq s, t \neq u)$ and for all $\lambda \in \mathbb{K}$ we have*

$$X_{tu} V_{rs}(\lambda) \subseteq V_{rs}(\lambda + 2) + V_{rs}(\lambda) + V_{rs}(\lambda - 2).$$

We are using Notation 3.2.

Proof. The above inclusion holds for each row of the table in Theorem 3.3. ∎

Theorem 3.5 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Let r, s, t denote mutually distinct elements in $\mathbb{I}$. Then for all $\lambda \in \mathbb{K}$ we have*

$$V_{rs}(\lambda) + V_{rs}(\lambda - 2) + \cdots = V_{rt}(\lambda) + V_{rt}(\lambda - 2) + \cdots. \tag{12}$$

We are using Notation 3.2.

Proof. Let Y denote the left-hand side of (12) and let Z denote the right-hand side of (12). We show $Y = Z$. We first show $Y \subseteq Z$. Since the dimension of V is finite, there exists a nonnegative integer j such that $V_{rs}(\lambda - 2i) = 0$ and $V_{rt}(\lambda - 2i) = 0$ for all integers $i > j$. Observe that $Y = \sum_{i=0}^{j} V_{rs}(\lambda - 2i)$ and $Z = \sum_{i=0}^{j} V_{rt}(\lambda - 2i)$. By (11) we find Z is the set of vectors in V on which

$$\prod_{i=0}^{j} (X_{rt} - (\lambda - 2i)I) \tag{13}$$

vanishes. Using the table in Theorem 3.3 (row $t = r, u \neq s$) we find that (13) vanishes on $V_{rs}(\lambda - 2i)$ for $0 \leq i \leq j$. Therefore $V_{rs}(\lambda - 2i) \subseteq Z$ for $0 \leq i \leq j$ so $Y \subseteq Z$.

To get $Z \subseteq Y$ interchange the roles of s, t in the argument so far. We conclude that $Y = Z$ and the result follows. ∎

Corollary 3.6 *Let V denote a finite-dimensional irreducible $\boxtimes$-module and fix $\lambda \in \mathbb{K}$. Then for distinct $r, s \in \mathbb{I}$ the dimension of $V_{rs}(\lambda)$ is independent of r, s. We are using Notation 3.2.*

Proof. We define a binary relation on the generators of $\boxtimes$ called *adjacent*. By definition two distinct generators X_{rs}, X_{tu} are adjacent whenever $r = t$ or $s = u$. We observe the adjacency relation is symmetric. We further observe that the generators of $\boxtimes$ are connected with respect to adjacency. By these comments it suffices to show $V_{rs}(\lambda), V_{tu}(\lambda)$ have the same dimension for adjacent generators X_{rs}, X_{tu}. First assume $r = t$. By Theorem 3.5 we find

$$V_{rs}(\lambda) + V_{rs}(\lambda - 2) + \cdots = V_{tu}(\lambda) + V_{tu}(\lambda - 2) + \cdots. \tag{14}$$

Applying Theorem 3.5 (with λ replaced by $\lambda - 2$), we find

$$V_{rs}(\lambda-2) + V_{rs}(\lambda-4) + \cdots = V_{tu}(\lambda-2) + V_{tu}(\lambda-4) + \cdots. \tag{15}$$

Let H denote the sum on either side of (15). Comparing (14) and (15) we find

$$V_{rs}(\lambda) + H = V_{tu}(\lambda) + H. \tag{16}$$

The sum on either side of (16) is direct, so $V_{rs}(\lambda)$ and $V_{tu}(\lambda)$ have the same dimension.

Next we assume $s = u$. Applying what we have done so far with (r, s, t, u, λ) replaced by $(s, r, u, t, -\lambda)$ we find $V_{sr}(-\lambda)$ and $V_{ut}(-\lambda)$ have the same dimension. Recall that $X_{sr} = -X_{rs}$ and $X_{ut} = -X_{tu}$ so $V_{sr}(-\lambda) = V_{rs}(\lambda)$ and $V_{ut}(-\lambda) = V_{tu}(\lambda)$. By these comments $V_{rs}(\lambda)$ and $V_{tu}(\lambda)$ have the same dimension. We have now shown $V_{rs}(\lambda), V_{tu}(\lambda)$ have the same dimension for adjacent generators X_{rs}, X_{tu} and the result follows. ∎

Corollary 3.7 *Let V denote a finite-dimensional irreducible $\boxtimes$-module and fix $\lambda \in \mathbb{K}$. Then for distinct $r, s \in \mathbb{I}$ the spaces $V_{rs}(\lambda), V_{rs}(-\lambda)$ have the same dimension. We are using Notation 3.2.*

Proof. By Corollary 3.6 we find $V_{rs}(\lambda), V_{sr}(\lambda)$ have the same dimension. But $X_{rs} = -X_{sr}$ so $V_{sr}(\lambda) = V_{rs}(-\lambda)$. ∎

Theorem 3.8 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Then the following $(i), (ii)$ hold.*

(i) For distinct $r, s \in \mathbb{I}$ the generator X_{rs} is diagonalizable on V.

(ii) There exists an integer $d \geq 0$ such that for all distinct $r, s \in \mathbb{I}$ the eigenvalues for X_{rs} on V are $d, d-2, \ldots, -d$.

Proof. Throughout this proof we will use Notation 3.2. Let r, s denote distinct elements of $\mathbb{I}$. Since $\mathbb{K}$ is algebraically closed, all the eigenvalues for X_{rs} on V are contained in $\mathbb{K}$. Since V has positive finite dimension, the action of X_{rs} on V has at least one eigenvalue λ. Since $\mathrm{Char}(\mathbb{K}) = 0$, the scalars $\lambda, \lambda+2, \ldots$ are mutually distinct, so they cannot all be eigenvalues for X_{rs} on V; consequently there exists $\theta \in \mathbb{K}$ such that $V_{rs}(\theta) \neq 0$ but $V_{rs}(\theta+2) = 0$. Similarly the scalars $\theta, \theta-2, \ldots$ are mutually distinct, so they cannot all be eigenvalues for X_{rs} on V; consequently there exists an integer $d \geq 0$ such that $V_{rs}(\theta - 2i)$ is nonzero for $0 \leq i \leq d$ and zero for $i = d+1$. We will now show that

$$V_{rs}(\theta) + \cdots + V_{rs}(\theta - 2d) \tag{17}$$

is equal to V.

By Corollary 3.4 and since $V_{rs}(\theta+2) = 0$, $V_{rs}(\theta - 2d - 2) = 0$ we find that (17) is X_{tu}-invariant for all distinct $t, u \in \mathbb{I}$. Therefore (17) is $\boxtimes$-invariant. Recall the $\boxtimes$-module V is irreducible so (17) is equal to either 0 or V. By construction each term in (17) is nonzero and there is at least one term, so (17) is nonzero. Therefore (17) is equal to V. This shows that the action of X_{rs} on V is diagonalizable with eigenvalues $\Delta = \{\theta - 2i | 0 \leq i \leq d\}$.

It remains to show $\theta = d$. By Corollary 3.7 we find $\Delta = -\Delta$. It follows that $\theta = -\theta + 2d$ so $\theta = d$. ∎

Definition 3.9 Let V denote a finite-dimensional irreducible $\boxtimes$-module. By the *diameter* of V we mean the nonnegative integer d from Theorem 3.8(ii).

Our discussion of $\boxtimes$-modules will continue after we recall the notion of a flag.

4 Flags

Definition 4.1 Let V denote a vector space over $\mathbb{K}$ with positive finite dimension. Let d denote a nonnegative integer. By a *flag on V of diameter d*, we mean a sequence $F_0, F_1, \ldots, F_d$ consisting of mutually distinct subspaces of V such that $F_0 \neq 0$, $F_{i-1} \subseteq F_i$ for $1 \leq i \leq d$, and $F_d = V$. We call F_i the i^{th} *component* of the flag.

Let V denote a vector space over $\mathbb{K}$ with positive finite dimension. Let d denote a nonnegative integer. By a *decomposition of V of diameter d*, we mean a sequence $V_0, V_1, \ldots, V_d$ consisting of nonzero subspaces of V such that

$$V = V_0 + \cdots + V_d \qquad \text{(direct sum)}.$$

We do not assume each of $V_0, \ldots, V_d$ has dimension 1. For $0 \leq i \leq d$ we call V_i the i^{th} *subspace* of the decomposition.

The following construction yields a flag on V. Let $V_0, \ldots, V_d$ denote a decomposition of V. Set

$$F_i = V_0 + \cdots + V_i$$

for $0 \leq i \leq d$. Then the sequence $F_0, \ldots, F_d$ is a flag on V. We say this flag is *induced* by $V_0, \ldots, V_d$.

Let $V_0, \ldots, V_d$ denote a decomposition of V. By the *inversion* of this decomposition, we mean the decomposition $V_d, \ldots, V_0$ of V.

We now discuss the notion of *opposite* flags. Let F and G denote flags on V. These flags are said to be *opposite* whenever there exists a decomposition $V_0, \ldots, V_d$ of V such that F is induced by $V_0, \ldots, V_d$ and G is induced by $V_d, \ldots, V_0$. In this case

$$F_i \cap G_j = 0 \qquad \text{if } i + j < d \tag{18}$$

and

$$V_i = F_i \cap G_{d-i} \qquad \text{for } 0 \leq i \leq d. \tag{19}$$

In particular $V_0, \ldots, V_d$ is uniquely determined by the ordered pair F, G. We say $V_0, \ldots, V_d$ is *induced* by F, G.

5 Finite-Dimensional Irreducible ⊠-Modules, Revisited

We return our attention to a finite-dimensional irreducible ⊠-module V. Recall the set $\mathbb{I}$ from Definition 1.1 has four elements. With each element of $\mathbb{I}$ we associate a flag on V. We start with a definition.

Definition 5.1 Let V denote a finite-dimensional irreducible ⊠-module. Let (r, s) denote an ordered pair of distinct elements of $\mathbb{I}$. By Theorem 3.8 the sequence $V_{rs}(-d), V_{rs}(2-d), \ldots, V_{rs}(d)$ is a decomposition of V, where d denotes the diameter. We will call this the *decomposition of V associated with* (r, s).

Note 5.2 For distinct $r, s \in \mathbb{I}$ the decomposition of V associated with (s, r) is the inversion of the decomposition of V associated with (r, s).

Lemma 5.3 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Let r, s denote distinct elements of $\mathbb{I}$ and consider the decomposition of V associated with (r, s). Then the flag induced by this decomposition is independent of s.*

Proof. Let t denote an element of $\mathbb{I}$ such that r, s, t are mutually distinct. It suffices to show that the flag induced by the decomposition of V associated with (r, s) equals the flag induced by the decomposition of V associated with (r, t). Let d denote the diameter of V. For $0 \leq i \leq d$ the i^{th} component of the first flag is $V_{rs}(-d) + \cdots + V_{rs}(2i - d)$ and the i^{th} component of the second flag is $V_{rt}(-d) + \cdots + V_{rt}(2i - d)$. These components are equal by Theorem 3.5 (with $\lambda = 2i - d$). Therefore the flags are equal. ∎

Definition 5.4 Let V denote a finite-dimensional irreducible $\boxtimes$-module. For $r \in \mathbb{I}$, by the *flag on V associated with r* we mean the flag discussed in Lemma 5.3.

The next corollary restates Lemma 5.3 in light of Definition 5.4.

Corollary 5.5 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. For $r \in \mathbb{I}$ consider the flag on V associated with r. The components of this flag are described as follows. Let d denote the diameter of V and pick $s \in \mathbb{I}$ such that $r \neq s$. Then for $0 \leq i \leq d$ the i^{th} component of the flag is*

$$V_{rs}(-d) + V_{rs}(2 - d) + \cdots + V_{rs}(2i - d). \tag{20}$$

Lemma 5.6 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Recall from Definition 1.1 that $\mathbb{I}$ has four elements and consider the corresponding flags on V from Definition 5.4. These four flags are mutually opposite.*

Proof. For distinct $r, s \in \mathbb{I}$ let F denote the flag on V associated with r and let G denote the flag on V associated with s. We show F and G are opposite. By Note 5.2 the decomposition of V associated with (r, s) is the inversion of the decomposition of V associated with (s, r). The first decomposition induces F and the second decomposition induces G. It follows that F and G are opposite. ∎

Lemma 5.7 *Let V denote a finite-dimensional irreducible $\boxtimes$-module of diameter d. Let r, s denote distinct elements of $\mathbb{I}$. For $0 \leq i \leq d$ the subspace $V_{rs}(2i - d)$ is the intersection of the following two sets:*

(i) The i^{th} component of the flag on V associated with r.

(ii) The $(d - i)^{th}$ component of the flag on V associated with s.

Proof. By Definition 5.4 the flag on V associated with r is induced by $V_{rs}(-d), \ldots, V_{rs}(d)$ and the flag on V associated with s is induced by $V_{rs}(d), \ldots, V_{rs}(-d)$. The result follows. ∎

6 From $\boxtimes$-Modules to $\mathcal{O}$-Modules

In this section we prove Theorem 1.7. We begin with a lemma.

Lemma 6.1 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Let r, s, t, u denote mutually distinct elements of $\mathbb{I}$. Let W denote a nonzero subspace of V such that $X_{rs}W \subseteq W$, $X_{tu}W \subseteq W$. Then $W = V$.*

Proof. Without loss of generality, we assume W is irreducible as a module for X_{rs}, X_{tu}. To show $W = V$, it suffices to show that W is $\boxtimes$-invariant. To this end, we show

$$X_{ab}W \subseteq W \tag{21}$$

for all distinct $a, b \in \mathbb{I}$. Assume ab is not one of rs, tu, sr, ut; otherwise (21) holds by the construction and (1). Replacing an appropriate subset of $\{rs, tu, ab\}$ by the corresponding subset of $\{sr, ut, ba\}$ we may assume, without loss of generality, that $a = s$ and $b = t$. Therefore it suffices to show $X_{st}W \subseteq W$. We define $W' := \{w \in W | X_{st}w \in W\}$ and show $W' = W$. By (2) we routinely find that $X_{rs}W' \subseteq W'$ and $X_{tu}W' \subseteq W'$. By these comments and the irreducibility of W we find either $W' = 0$ or $W' = W$.

Claim. $W' \neq 0$.

Proof of claim. Let d denote the diameter of V. Define $W_j = W \cap V_{rs}(2j - d)$ for $0 \leq j \leq d$. The nonzero spaces among $W_0, \ldots, W_d$ are eigenspaces of X_{rs} on W. By Theorem 3.8 we find $W = \sum_{j=0}^{d} W_j$. By this and since $W \neq 0$ we find $W_0, \ldots, W_d$ are not all 0. Define $m = \min\{j | 0 \leq j \leq d, W_j \neq 0\}$.

Define $W_j^* = W \cap V_{tu}(2j - d)$ for $0 \leq j \leq d$. The nonzero spaces among $W_0^*, \ldots, W_d^*$ are eigenspaces of X_{tu} on W. By Theorem 3.8 we find $W = \sum_{j=0}^{d} W_j^*$. By this and since $W \neq 0$ we find $W_0^*, \ldots, W_d^*$ are not all 0. Define $n = \min\{j | 0 \leq j \leq d, W_j^* \neq 0\}$.

We show $m = n$. To do this we first assume $m < n$ and get a contradiction. By construction $W_j^* \subseteq V_{tu}(2j - d)$ for $0 \leq j \leq d$ and $W = \sum_{j=n}^{d} W_j^*$ so $W \subseteq \sum_{j=n}^{d} V_{tu}(2j - d)$. Therefore W is contained in the $(d - n)^{th}$ component of the flag on V associated with u. By construction $W_m \subseteq V_{rs}(2m - d)$ so $W_m \subseteq \sum_{j=0}^{m} V_{rs}(2j - d)$. Therefore W_m is contained in the m^{th} component of the flag on V associated with r. Recall by Lemma 5.6 that the flag on V associated with u and the flag on V associated with r are opposite. By (18) the $(d - n)^{th}$ component of the flag on V associated with u and the m^{th} component of the flag on V associated with r have 0 intersection. Therefore $W \cap W_m = 0$, contradicting the fact that W_m is nonzero in W. Therefore $m \geq n$.

Next we assume $m > n$ and get a contradiction. By construction $W_j \subseteq V_{rs}(2j - d)$ for $0 \leq j \leq d$ and $W = \sum_{j=m}^{d} W_j$ so $W \subseteq \sum_{j=m}^{d} V_{rs}(2j - d)$. Therefore W is contained in the $(d - m)^{th}$ component of the flag on V associated with s. By construction $W_n^* \subseteq V_{tu}(2n - d)$ so $W_n^* \subseteq \sum_{j=0}^{n} V_{tu}(2j - d)$. Therefore W_n^* is contained in the n^{th} component of the flag on V associated with t. Recall by Lemma 5.6 that the flag on V associated with s and the flag on V associated with t are opposite. By (18) the $(d - m)^{th}$ component of the flag on V associated with s and the n^{th} component of the flag on V associated with t have 0 intersection. Therefore $W \cap W_n^* = 0$, contradicting the fact that W_n^* is nonzero in W. Therefore $m \leq n$.

Combining the previous two paragraphs we find $m = n$. By the previous paragraph we find W_n^* is contained in the intersection of the $(d-m)^{th}$ component of the flag on V associated with s and the m^{th} component of the flag on V associated with t. By Lemma 5.7 we find $W_n^* \subseteq V_{st}(d-2m)$. Since $V_{st}(d-2m)$ is an eigenspace for X_{st} we find $X_{st}W_n^* \subseteq W_n^*$. Therefore $W_n^* \subseteq W'$, so $W' \neq 0$ and the claim is proved.

This shows that $W' = W$ and therefore $X_{st}W \subseteq W$. By this and the comments prior to the claim we have now shown that $X_{ab}W \subseteq W$ for all distinct $a, b \in \mathbb{I}$. Therefore W is $\boxtimes$-invariant. Since the $\boxtimes$-module V is irreducible we have $W = V$. ∎

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. Since X_{01}, X_{23} satisfy (4) and (5) there exists an $\mathcal{O}$-module structure on V such that X, Y act on V as X_{01}, X_{23} respectively. This structure is unique since X, Y generate $\mathcal{O}$. This structure is irreducible by Lemma 6.1. This structure has type $(0,0)$ by Theorem 3.8(ii). ∎

7 From $\mathcal{O}$-Modules to $\boxtimes$-Modules

In this section we prove Theorem 1.8. We refer to the following setup.

Definition 7.1 Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0,0)$. Let A, A^* denote the actions on V of the generators X, Y respectively and recall by Corollary 2.7 that A, A^* is a tridiagonal pair on V. Let $\mathbb{I}$ denote the set consisting of the four symbols $\{0, 1, 2, 3\}$. With each element of $\mathbb{I}$ we will associate a flag on V of diameter d where d denotes the diameter of V. For $0 \leq i \leq d$ the i^{th} component of this flag is given as follows.

Element of $\mathbb{I}$	i^{th} component of associated flag
0	$V_A(-d) + \cdots + V_A(2i-d)$
1	$V_A(d) + \cdots + V_A(d-2i)$
2	$V_{A^*}(-d) + \cdots + V_{A^*}(2i-d)$
3	$V_{A^*}(d) + \cdots + V_{A^*}(d-2i)$

Lemma 7.2 *[16, Theorem 7.3] The four flags from Definition 7.1 are mutually opposite.*

Definition 7.3 Adopt the assumptions of Definition 7.1. For distinct $r, s \in \mathbb{I}$ let $x_{rs} : V \to V$ denote the unique linear transformation satisfying the following conditions. By Definition 7.1 each of r, s is associated with a flag on V of diameter d. By Lemma 7.2 the flags r, s are opposite; let $U_0, \ldots, U_d$ denote the induced decomposition of V. For $0 \leq i \leq d$, U_i is an eigenspace for x_{rs} with eigenvalue $2i - d$.

We will be using the following notation.

Notation 7.4 With reference to Definition 7.3, for distinct $r, s \in \mathbb{I}$ and for all $\lambda \in \mathbb{K}$ we define

$$V_{rs}(\lambda) = \{v \in V | x_{rs}v = \lambda v\}.$$

The next two lemmas illustrate the notation in Definition 7.3 and will be useful later in the paper.

Lemma 7.5 *With reference to Definition 7.3, we have $x_{01} = A$ and $x_{23} = A^*$.*

Proof. Immediate from Definition 7.3. ∎

Lemma 7.6 *With reference to Definition 7.3, for $r \in \mathbb{I}$ consider the flag on V associated with r. The components of this flag are described as follows. Pick $s \in \mathbb{I}$ such that $r \neq s$. Then for $0 \leq i \leq d$ the i^{th} component of the flag is*

$$V_{rs}(-d) + \cdots + V_{rs}(2i - d).$$

We are using Notation 7.4.

Proof. By Definition 7.3 the decomposition $V_{rs}(-d), \ldots, V_{rs}(d)$ of V is induced by the flag on V associated with r and the flag on V associated with s. By the discussion at the end of Section 4 the flag r is induced by $V_{rs}(-d), \ldots, V_{rs}(d)$. The result follows. ∎

We will need the following three lemmas.

Lemma 7.7 *With reference to Definition 7.3, for distinct $r, s \in \mathbb{I}$ we have*

$$x_{rs} + x_{sr} = 0.$$

Proof. Let $U_0, \ldots, U_d$ denote the decomposition induced by the flag on V associated with r and the flag on V associated with s. By Definition 7.3 and with reference to Notation 7.4, we find $V_{rs}(2i - d) = U_i = V_{sr}(d - 2i)$ for $0 \leq i \leq d$. The result follows. ∎

Lemma 7.8 *With reference to Definition 7.3, for mutually distinct $r, s, t \in \mathbb{I}$ we have*

$$[x_{rs}, x_{st}] = 2x_{rs} + 2x_{st}. \tag{22}$$

Proof. Throughout this proof we will use Notation 7.4.

We break our proof into the following three cases:

(i) st is one of $01, 10, 23, 32$.

(ii) rs is one of $01, 10, 23, 32$.

(iii) Neither rs nor st is one of $01, 10, 23, 32$.

Case (i): We will invoke Lemma 3.1. We begin with some comments. Recall that x_{rs} is diagonalizable on V with eigenvalues $d, d-2, \ldots, -d$. We now show that

$$(x_{st} + (2i - d)I)V_{rs}(2i - d) \subseteq V_{rs}(2i - d + 2) \tag{23}$$

for $0 \leq i \leq d$. Let u denote the unique element of $\mathbb{I}$ such that r, s, t, u are mutually distinct. By Lemma 7.5 and Lemma 7.7 we find that x_{st}, x_{ur} is either $\epsilon A, \epsilon^* A^*$ or $\epsilon^* A^*, \epsilon A$ for some $\epsilon, \epsilon^* \in \{1, -1\}$. By this and Definition 7.1 we find x_{st}, x_{ur} is a tridiagonal pair on V of diameter d and $d, d-2, \ldots, -d$ is both an eigenvalue sequence and dual eigenvalue sequence of x_{st}, x_{ur}. By Lemma 7.6 and using [10, Theorem 4.6] we find that (23) holds for $0 \leq i \leq d$. By this and Lemma 3.1 we find (22) holds.

Case (ii): Observe that sr is one of $01, 10, 23, 32$. By case (i) of the current proof we find that

$$[x_{ts}, x_{sr}] = 2x_{ts} + 2x_{sr}. \tag{24}$$

By Lemma 7.7 we find $x_{ts} = -x_{st}$ and $x_{sr} = -x_{rs}$. Substituting this into (24) and simplifying the result we find that (22) holds.

Case (iii): Similarly to case (i) we will invoke Lemma 3.1. Recall that x_{rs} is diagonalizable on V with eigenvalues $d, d-2, \ldots, -d$. We now show that

$$(x_{st} + (2i-d)I)V_{rs}(2i-d) \subseteq V_{rs}(2i-d+2) \tag{25}$$

for $0 \leq i \leq d$. Observe rt is one of $01, 10, 23, 32$. By case (ii) of the current proof we find that $[x_{rt}, x_{ts}] = 2x_{rt} + 2x_{ts}$. By Lemma 3.1 (with $\lambda = d - 2i$) and since $x_{rt} = -x_{tr}$, $x_{ts} = -x_{st}$ we find

$$(x_{st} + (2i-d)I)V_{tr}(2i-d) \subseteq V_{tr}(2i-d-2) \qquad 0 \leq i \leq d. \tag{26}$$

For $0 \leq i \leq d$ we have

$$\begin{aligned}
(x_{st} + (2i-d)I)V_{rs}(2i-d) &\subseteq (x_{st} + (2i-d)I)\sum_{n=0}^{i} V_{rs}(2n-d) && \\
&= (x_{st} + (2i-d)I)\sum_{n=0}^{i} V_{tr}(d-2n) && \text{by Lemma 7.6} \\
&\subseteq \sum_{n=0}^{i+1} V_{tr}(d-2n) && \text{by (26)} \\
&= \sum_{n=0}^{i+1} V_{rs}(2n-d) && \text{by Lemma 7.6}
\end{aligned}$$

and

$$\begin{aligned}
(x_{st} + (2i-d)I)V_{rs}(2i-d) &\subseteq (x_{st} + (2i-d)I)\sum_{n=i}^{d} V_{rs}(2n-d) && \\
&= (x_{st} + (2i-d)I)\sum_{n=i}^{d} V_{st}(d-2n) && \text{by Lemma 7.6} \\
&= \sum_{n=i+1}^{d} V_{st}(d-2n) && \\
&= \sum_{n=i+1}^{d} V_{rs}(2n-d) && \text{by Lemma 7.6.}
\end{aligned}$$

Combining these observations we find that (25) holds for $0 \leq i \leq d$. By this and Lemma 3.1 we find that (22) holds. ∎

Lemma 7.9 *With reference to Definition 7.3, for mutually distinct $r, s, t, u \in \mathbb{I}$ we have*

$$[x_{rs}, [x_{rs}, [x_{rs}, x_{tu}]]] = 4[x_{rs}, x_{tu}]. \tag{27}$$

Proof. Throughout this proof we will use Notation 7.4.

We will invoke Lemma 2.1. We begin with some comments. Recall that x_{rs} is diagonalizable on V with eigenvalues $d, d-2, \ldots, -d$. We now show that

$$x_{tu}V_{rs}(2i-d) \subseteq V_{rs}(2i-d+2) + V_{rs}(2i-d) + V_{rs}(2i-d-2) \tag{28}$$

for $0 \le i \le d$. By Lemma 7.8 we find $[x_{st}, x_{tu}] = 2x_{st} + 2x_{tu}$. By this and Lemma 3.1 we find

$$(x_{tu} + (2i-d)I)V_{st}(2i-d) \subseteq V_{st}(2i-d+2) \qquad 0 \le i \le d. \tag{29}$$

By Lemma 7.8 we find $[x_{ru}, x_{ut}] = 2x_{ru} + 2x_{ut}$. By Lemma 3.1 (with $\lambda = d - 2i$) and since $x_{ru} = -x_{ur}$, $x_{ut} = -x_{tu}$ we find

$$(x_{tu} + (2i-d)I)V_{ur}(2i-d) \subseteq V_{ur}(2i-d-2) \qquad 0 \le i \le d. \tag{30}$$

For $0 \le i \le d$ we have

$$\begin{aligned}
x_{tu}V_{rs}(2i-d) &\subseteq x_{tu}\sum_{n=i}^{d} V_{rs}(2n-d) \\
&= x_{tu}\sum_{n=i}^{d} V_{st}(d-2n) && \text{by Lemma 7.6} \\
&\subseteq \sum_{n=i-1}^{d} V_{st}(d-2n) && \text{by (29)} \\
&= \sum_{n=i-1}^{d} V_{rs}(2n-d) && \text{by Lemma 7.6}
\end{aligned}$$

and

$$\begin{aligned}
x_{tu}V_{rs}(2i-d) &\subseteq x_{tu}\sum_{n=0}^{i} V_{rs}(2n-d) \\
&= x_{tu}\sum_{n=0}^{i} V_{ur}(d-2n) && \text{by Lemma 7.6} \\
&\subseteq \sum_{n=0}^{i+1} V_{ur}(d-2n) && \text{by (30)} \\
&= \sum_{n=0}^{i+1} V_{rs}(2n-d) && \text{by Lemma 7.6.}
\end{aligned}$$

Combining these observations we find that (28) holds for all $0 \le i \le d$. By this and Lemma 2.1 we find that (27) holds. ∎

Theorem 7.10 *With reference to Definition 7.3, there exists a unique $\boxtimes$-module structure on V such that the generator X_{rs} acts on V as x_{rs} for all distinct $r, s \in \mathbb{I}$. This $\boxtimes$-module structure is irreducible.*

Proof. Using Lemma 7.7, Lemma 7.8, and Lemma 7.9 we find that there exists a $\boxtimes$-module structure on V such that X_{rs} acts on V as x_{rs} for distinct $r, s \in \mathbb{I}$. This module structure is unique since the set $\{X_{rs} | r, s \in \mathbb{I}, r \neq s\}$ is a generating set for $\boxtimes$. By Lemma 7.5 we find that A, A^* are among the linear transformations x_{rs}, $r, s \in \mathbb{I}$, $r \neq s$. Recall that A, A^* is a tridiagonal pair on V. Therefore the $\boxtimes$-module V is irreducible by Definition 2.2(iv). ∎

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. Consider the $\boxtimes$-module structure on V from Theorem 7.10. In this module structure X_{01}, X_{23} act on V as X, Y respectively by Definition 7.1 and Lemma 7.5. Next we show that this $\boxtimes$-module structure is unique. Suppose we are given any $\boxtimes$-module structure on V where X_{01}, X_{23} act on V as X, Y respectively. This module structure is irreducible since the $\mathcal{O}$-module V is irreducible. For each generator X_{rs} of $\boxtimes$ the action on V is determined by the decomposition of V associated with (r, s). By Lemma 5.7 the decomposition of V associated with (r, s) is determined by the flag on V associated with r and the flag on V associated with s. Therefore our $\boxtimes$-module structure on V is determined by the four flags associated with the four elements of $\mathbb{I}$. By Corollary 5.5 the four flags associated with the four elements of $\mathbb{I}$ are determined by the actions of X_{01} and X_{23} on V. Therefore the given $\boxtimes$-module structure on V is determined by the actions of X_{01} and X_{23} on V, so the $\boxtimes$-module structure is unique. We have now shown there exists a unique $\boxtimes$-module structure on V where X_{01}, X_{23} act on V as X, Y respectively. We mentioned earlier that this module structure is irreducible. ∎

Acknowledgements: This paper was written while the author was a graduate student at the University of Wisconsin-Madison. The author would like to thank his advisor Paul Terwilliger for his many valuable ideas and suggestions.

References

[1] H. Au-Yang, B.M. McCoy, J.H.H. Perk, S. Tang, and M.L. Yan. Commuting transfer matrices in chiral Potts modules: Solutions of the star-triangle equations with genus greater than 1. *Phys. Lett. A* **123** (1987) 219-223.

[2] G. Benkart and P. Terwilliger. Irreducible modules for the quantum affine algebra $U_q(\widehat{sl}_2)$ and its Borel subalgebra. *J. Algebra* **282** (2004), no. 1, 172–194.

[3] E. Date and S.S Roan. The structure of quotients of the Onsager algebra by closed ideals. *J. Phys. A: Math. Gen.* **33** (2000) 3275–3296.

[4] B. Davies. Onsager's algebra and superintegrability. *J. Phys. A:Math. Gen.* **23** (1990), 2245–2261.

[5] B. Davies. Onsager's algebra and the Dolan-Grady condition in the non-self-dual case. *J. Math. Phys.* **32** (1991), 2945–2950.

[6] L. Dolan and M. Grady. Conserved charges from self-duality. *Phys. Rev. D* **25** (1982), 1587-1604.

[7] G. von Gehlen and R. Rittenberg. $\mathbb{Z}_n$-symmetric quantum chains with infinite set of conserved charges and $\mathbb{Z}_n$ zero modes. *Nucl. Phys. B* **257** (1985), 351-370.

[8] B. Hartwig and P. Terwilliger. The Tetrahedrom Algebra, the Onsager algebra, and the $\mathfrak{sl}_2$ loop algebra. *J. Algebra*, submitted.

[9] T. Ito and P. Terwilliger. Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl}_2)$. *Ramanujan J.*, to appear.

[10] T. Ito, K. Tanabe, and P. Terwilliger. Some algebra related to P- and Q-polynomial association schemes. *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2000.

[11] T. Ito and P. Terwilliger. The shape of a tridiagonal pair. *J. Pure Appl. Algebra,* **188** (2004) 145–160.

[12] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2),* **65** (1944) 117–149.

[13] J.H.H. Perk. Star-triangle relations, quantum Lax pairs, and higher genus curves. *Proceedings of Symposia in Pure Mathematics* **49** 341–354. Amer. Math. Soc., Providence, RI, 1989.

[14] S. S. Roan. Onsager's algebra, loop algebra and chiral Potts model. Preprint MPI 91-70, Max Plank Institute for Mathematics, Bonn, 1991.

[15] P. Terwilliger. Two relations that generalize the q-Serre and Dolan-Grady relations. Proceedings of Nagoya 1999 Workshop on Physics and Combinatorics, Nagoya, Japan 1999, World ScientificPublishing Co., Inc., River Edge, NJ, Providence, RI, 2000.

[16] P. Terwilliger. Leonard pairs from 24 points of view. *Rocky Mountain J. Math.* **32** (2) (2002) 827–888.

Brian Hartwig
Department of Mathematics
University of Wisconsin
480 Lincoln Drive
Madison, Wisconsin, 53706, USA
email: hartwig@math.wisc.edu